

Homework 1 Solution Key

Math 121A, Fall 2025

September 6, 2025

Problem 1. Section 1.1

- (a) Problem 12
- (b) Problem 15
- (c) Problem 16

Problem 2. In the following problems, find the limit of the given sequence as $n \rightarrow \infty$.

(a)

$$\lim_{n \rightarrow \infty} \frac{(n+1)^2}{\sqrt{3+5n^2+4n^4}} = \lim_{n \rightarrow \infty} \frac{1/n^2}{1/n^2} \frac{n^2+2n+1}{\sqrt{3+5n^2+4n^4}} = \lim_{n \rightarrow \infty} \frac{1+2n/n^2+1/n^2}{\sqrt{3/n^4+5n^2/n^2+4}} = \frac{1}{\sqrt{4}} = \frac{1}{2}.$$

(b) Both 2^n and n^2 go to infinity, so we may use L'Hopital's rule (actually, we will use it twice:

$$\lim_{n \rightarrow \infty} \frac{2^n}{n^2} = \lim_{n \rightarrow \infty} \frac{\log(2)2^n}{2n} = \lim_{n \rightarrow \infty} \frac{(\log(2))^2 2^n}{2} = \infty.$$

(c) Just like part (b), but the reciprocal:

$$\lim_{n \rightarrow \infty} \frac{n^2}{2^n} = \lim_{n \rightarrow \infty} \frac{2n}{\log(2)2^n} = \lim_{n \rightarrow \infty} \frac{2}{(\log(2))^2 2^n} = 0.$$

(d) Observe that if $n > 8 \geq 2e$, then $\frac{e}{n} \leq \frac{1}{2}$, so

$$\frac{e^n}{n!} = \frac{e}{n} \frac{e}{n-1} \cdots \frac{e}{3} \frac{e}{2} \frac{e}{1} \leq \frac{1}{2} \frac{e^{(n-1)}}{(n-1)!}.$$

This means that $\frac{e^n}{n!} \leq \left(\frac{1}{2}\right)^{n-4} \frac{e^4}{4!}$, and so

$$\lim_{n \rightarrow \infty} \frac{e^n}{n!} \leq \lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^{n-4} \frac{e^4}{4!} = 0.$$

(e) Observe that

$$\frac{n^n}{n!} = \frac{n}{n} \frac{n}{n-1} \frac{n}{n-2} \cdots \frac{n}{2} \frac{n}{1} \geq n,$$

so $\lim_{n \rightarrow \infty} \frac{n^n}{n!} \geq \lim_{n \rightarrow \infty} n = \infty$.

(f) Observe that

$$\lim_{n \rightarrow \infty} \log \left((1 + n^2)^{1/\log n} \right) = \lim_{n \rightarrow \infty} \frac{1}{\log n} \log(1 + n^2) = 2.$$

Thus, $\lim_{n \rightarrow \infty} (1 + n^2)^{1/\log n} = e^2$.

(g) Observe that

$$\lim_{n \rightarrow \infty} \frac{(n!)^2}{(2n)!} = \lim_{n \rightarrow \infty} \frac{n}{2n} \frac{n-1}{2n-1} \cdots \frac{1}{2n+1} \frac{n!}{n!} \leq \lim_{n \rightarrow \infty} \frac{1}{2n+1} = 0.$$

(h) Notice that $\lim_{n \rightarrow \infty} \cos(1/n) = 1$, so $\lim_{n \rightarrow \infty} n \cos(1/n) = \infty$.

Problem 3. Use the integral test to prove the so-called “ p -series test”. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \begin{cases} \text{convergent} & \text{if } p > 1 \\ \text{divergent} & \text{if } p \leq 1 \end{cases}.$$

Caution: do $p = 1$ separately.

Problem 4. Decide whether each of the following series are convergent or divergent.

- (a) $\sum_{n=1}^{\infty} \frac{2^n}{n}$.
- (b) $\sum_{n=1}^{\infty} \frac{2^n}{(2+\epsilon)^n}$, where $\epsilon > 0$ is a fixed positive number.
- (c) $\sum_{n=1}^{\infty} \frac{1}{\log n}$.
- (d) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$.
- (e) $\sum_{n=1}^{\infty} \frac{n-a}{n^{2b+1}+c}$, where $a, c \in \mathbb{R}$ and $b > 1$.
- (f) $\sum_{n=1}^{\infty} e^{-n^2}$. (*Hint: compare with another series, e.g. $\sum e^{-n}$.*)
- (g) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$.
- (h) $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$.
- (i) $\sum_{n=1}^{\infty} \frac{\sin|n|}{n^2}$.

HW 1 Solns

Problem 1: (section 1.1)

$$\begin{aligned} \text{a) (1.1.1.2)} \\ n=2 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\ &= \left(\sum_{n=0}^{\infty} \frac{1}{2^n} \right) - 1 \\ &= \frac{1}{1 - \frac{1}{2}} - 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} n=3 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{3^n} &= \frac{1}{1 - \frac{1}{3}} - 1 \\ &= \frac{1}{2} \end{aligned}$$

b) (1.1.1.5)



At each step, $\frac{1}{4}$ of the total filled area is removed. So the sequence of filled areas is

$$1, \frac{3}{4}, \left(\frac{3}{4}\right)^2, \dots, \left(\frac{3}{4}\right)^n, \dots$$

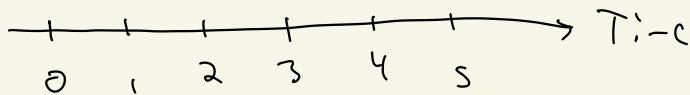
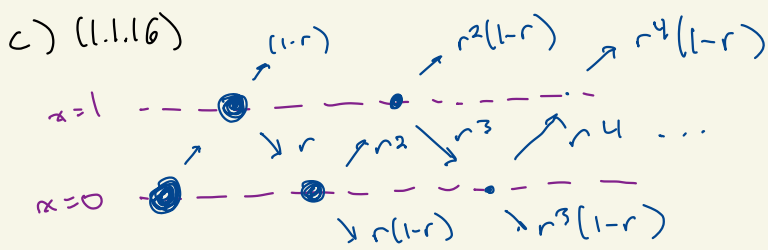
The sequence of removed area is

$$\frac{1}{4}, \frac{1}{4}\left(\frac{3}{4}\right), \frac{1}{4}\left(\frac{3}{4}\right)^2, \dots, \frac{1}{4}\left(\frac{3}{4}\right)^n$$

$$\text{Total Removed Area} = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{4} \frac{1}{1 - \frac{3}{4}} = 1.$$

This makes sense since the sequence of filled areas

$$\lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^n = 0.$$



- At odd timesteps (time = 1, 3, 5, ...) we repeat $\alpha=1$ and

Time: 1 3 5 ... $2k+1$

Escaping: $(1-r), r^2(1-r), r^4(1-r), \dots, r^{2k}(1-r)$

So, total # particles escaping is

$$\# \text{ Particles escaping at } t = (1-r) + r^2(1-r) + r^4(1-r) + \dots$$

$$= (1-r) \sum_{k=0}^{\infty} r^{2k}$$

$$= (1-r) \sum_{k=0}^{\infty} (r^2)^k$$

$$= (1-r) \left(\frac{1}{1-r^2} \right)$$

$$= \frac{1}{1+r}$$

Now, since $0 < r < 1$, $\lim_{t \rightarrow \infty} \left(\text{Particles in chamber} \right) = \lim_{t \rightarrow \infty} r^t = 0$

So all particles must eventually escape,

$$\# \text{ Particles escaping from } 0 = 1 - \left(\# \text{ particles escaping from } 1 \right)$$

$$= 1 - \frac{1}{1+r}$$

• To maximize this, since $\frac{1}{1+r}$ is decreasing,  the function $1 - \frac{1}{1+r}$ is increasing: 

the best we can do is pick r as large as possible:

So,

$$\text{Largest fraction particles escaping} = 1 - \frac{1}{1+r}$$

at $x=0$

$$\begin{aligned} & \text{1 - } \frac{1}{1+r} \text{ is increasing,} \\ & \text{and } 0 < r < 1. \rightarrow \leq 1 - \frac{1}{1+1} \\ & = \frac{1}{2} \end{aligned}$$

Problem 2

$$\begin{aligned} \text{a) } \lim_{n \rightarrow \infty} \frac{(n+1)^2}{\sqrt{3+5n^2+4n^4}} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} (n+1)^2}{\frac{1}{n^2} \sqrt{3+5n^2+4n^4}} \\ &= \lim_{n \rightarrow \infty} \frac{\left(\frac{n}{n} + \frac{1}{n}\right)^2}{\sqrt{\frac{3}{n^4} + \frac{5n^2}{n^4} + \frac{4n^4}{n^4}}} \\ &= \frac{1}{\sqrt{4}} \\ &= \frac{1}{2} \end{aligned}$$

b) Both 2^n and n^2 go to infinity, so we may use L'Hospital's rule:

$$\lim_{n \rightarrow \infty} \frac{2^n}{n^2} = \lim_{n \rightarrow \infty} \log 2 \frac{2^n}{2n} = \lim_{n \rightarrow \infty} (\log 2)^2 \frac{2^n}{2} = \boxed{\infty}$$

$\uparrow$ L'Hospital $\uparrow$ L'Hospital again

c) Just like the reciprocal of part b:

$$\lim_{n \rightarrow \infty} \frac{n^2}{2^n} = \lim_{n \rightarrow \infty} \frac{2}{(\log 2)^2 2^n} = \boxed{0}$$

d) Observe that if $n > 8 \geq 2e$, then $\frac{e}{n} \leq \frac{1}{2}$, so

$$\frac{e^n}{n!} = \frac{e}{n} \left(\frac{e}{n-1} \right) \cdots \frac{e}{3} \cdot \frac{e}{2} \cdot \frac{e}{1} \leq \frac{1}{2} \frac{e^{(n-1)}}{(n-1)!}$$

$$\text{So for } n > 8, \frac{e^n}{n!} \leq \left(\frac{1}{2} \right)^{n-4} \frac{e^4}{4!}.$$

$$\lim_{n \rightarrow \infty} \frac{e^n}{n!} \leq \lim_{n \rightarrow \infty} \left(\frac{1}{2} \right)^{n-4} \frac{e^4}{4!} = \boxed{0}$$

e) Observe that

$$\frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{(n-1)} \cdots \frac{n}{2} \cdot \frac{n}{1} \geq n,$$

$$\text{So } \lim_{n \rightarrow \infty} \frac{n^n}{n!} \geq \lim_{n \rightarrow \infty} n = \infty$$

$$\Rightarrow \boxed{\lim_{n \rightarrow \infty} \frac{n^n}{n!} = \infty}$$

f) Observe that

$$\lim_{n \rightarrow \infty} \log \left((1+n^2)^{1/\log n} \right) = \lim_{n \rightarrow \infty} \frac{1}{\log n} \log(1+n^2) = 2$$

$$\Rightarrow \boxed{\lim_{n \rightarrow \infty} (1+n^2)^{1/\log n} = e^2}$$

g) Observe that

$$\lim_{n \rightarrow \infty} \frac{(n!)^2}{(2n)!} = \lim_{n \rightarrow \infty} \frac{n}{2n} \frac{(n-1)}{(2n-1)} \cdot \dots \cdot \frac{1}{n+1} \frac{n!}{n!}$$
$$\leq \lim_{n \rightarrow \infty} \frac{1}{n+1}$$

$$= \boxed{0}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(n!)^2}{(2n)!} = 0.$$

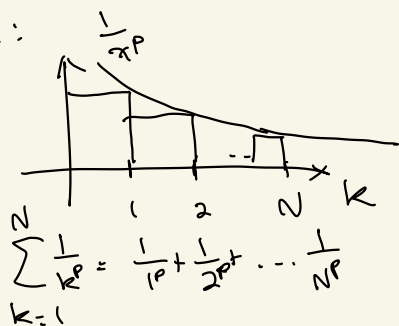
h) Since $\lim_{n \rightarrow \infty} \cos\left(\frac{1}{n}\right) = 1$, we have $\lim_{n \rightarrow \infty} n \cos\left(\frac{1}{n}\right) = \infty$.

Problem 3:

Integral test!

• If $p > 1$, then

Notice:



$$\sum_{k=1}^N \frac{1}{k^p} = \frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{N^p} \leq 1 + \int_1^N \frac{1}{x^p} dx$$

$$p > 1 \rightarrow = 1 - \frac{1}{p-1} \left(1 - \frac{1}{N^{p-1}} \right)$$

So

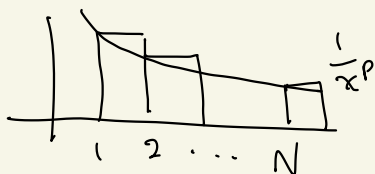
$$\lim_{N \rightarrow \infty} \sum_{k=1}^N \frac{1}{k^p} \leq \lim_{N \rightarrow \infty} 1 - \frac{1}{p-1} \left(1 - \frac{1}{N^{p-1}} \right)$$

$$= 1 - \frac{1}{p-1}$$

$\Rightarrow$ Converges for $p > 1$!

But if $0 < p \leq 1$

$$\sum_{k=1}^N \frac{1}{k^p}$$



$$\sum_{k=1}^N \frac{1}{k^p} \geq \int_1^{N+1} \frac{1}{x^p} dx = \begin{cases} \log(N+1) & p=1 \\ -\frac{1}{p-1} \left(1 - \frac{1}{(N+1)^{p-1}}\right) & p < 1 \end{cases}$$

So

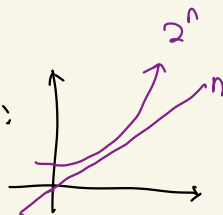
$$\lim_{N \rightarrow \infty} \sum_{k=1}^N \frac{1}{k^p} \geq \begin{cases} \lim_{N \rightarrow \infty} \log(N+1) = \infty \\ \lim_{N \rightarrow \infty} -\frac{1}{p-1} \left(1 - \frac{1}{(N+1)^{p-1}}\right) = \infty \end{cases}$$

$\Rightarrow$ Diverges when $p \leq 1$!

Problem 4:

a) If $n \geq 0$, then $2^n > n$ (look at graph:

$$\text{So } \sum_{n=1}^{\infty} \frac{2^n}{n} \geq \sum_{n=1}^{\infty} 1 = \infty$$



$\Rightarrow$ Diverges by comparison Test.

b) Since $0 < \frac{2}{2+\epsilon} < 1$, this is

$$\sum_{n=1}^{\infty} \left(\frac{2}{2+\epsilon}\right)^n \leftarrow \text{converges by geometric series}$$

c) Since $\log n \leq n$ for $n > 1$, $\frac{1}{n} \leq \frac{1}{\log n}$, so

$$\sum_{n=1}^{\infty} \frac{1}{\log n} \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty \leftarrow \text{Diverges by comparison with } p \text{ series, } p=1.$$

d) Notice that $\frac{1}{x \log x}$ is positive, decreasing when $x > 2$.

We can use the integral test, adjusting to

$$\text{Then } \int_2^{\infty} \frac{1}{x \log x} dx = \int_{\log 2}^{\infty} \frac{1}{u} du = (\log(\log 2)) - \log(\infty)$$

$u = \log x$
 $du = \frac{1}{x} dx$

$= \infty$

$\Rightarrow$ Diverges by integral test.

$$e) \sum_{n=1}^{\infty} \frac{n-a}{n^{2b+1+c}} = \underbrace{\sum_{n=1}^{\infty} \frac{n}{n^{2b+1+c}}}_{(*)} - \sum_{n=1}^{\infty} \frac{a}{n^{2b+1+c}} \quad (**)$$

$$\frac{n}{n^{2b+1+c}} = \frac{1}{n^{2b+1+\frac{c}{n}}} \xrightarrow{n \rightarrow \infty} \frac{1}{n^{2b}}$$

Since $b > 1$, $(*)$ and $(**)$ both converge, as they are (in large n limit) p -series.

f) Since e^x is increasing and $n^2 \geq n$ when $n \geq 1$,

$$e^{n^2} \geq e^n \Rightarrow \frac{1}{e^n} \geq \frac{1}{e^{n^2}}$$

$$\Rightarrow \sum_{n=1}^{\infty} e^{-n^2} \leq \sum_{n=1}^{\infty} \frac{1}{e^n} \leftarrow \text{converges by integral test: } \int_1^{\infty} e^{-x} dx = e^{-1}.$$

g) Just like Problem 2 part d),
when $n > 8$,

$$\frac{e^n}{n!} \leq \left(\frac{1}{2}\right)^{n-4} \frac{e^4}{4!}$$

so
$$\sum_{n=9}^{\infty} \frac{e^n}{n!} \leq \frac{e^4}{4!} \sum_{n=9}^{\infty} \left(\frac{1}{2}\right)^{n-4} \leftarrow \text{Converges by comparison with geometric series } r=1/2.$$

h)
$$\frac{(n!)^2}{(2n)!} = \frac{n}{2n} \frac{(n-1)}{(2n-1)} \cdots \frac{1}{2n-(n+1)} \frac{n!}{n!}$$

since $2(n-k) = 2n-2k \leq 2n-k$

$$\Rightarrow \frac{1}{2n-k} \leq \frac{1}{2(n-k)} = \frac{1}{2^n}$$

$$\leq \frac{n}{2n} \frac{(n-1)}{2(n-1)} \frac{(n-2)}{2(n-2)} \cdots \frac{1}{2(n-1)} \frac{n!}{n!}$$

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \leftarrow \text{Converges by comparison with geom. series.}$$

i) Since $|\sin n\pi| \leq 1$,

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\leftarrow$ Converges by comparison with p-series, $p=2$.